Brief Paper

Structural properties and poles assignability of LTI singular systems under output feedback

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Abstract

This paper studies the structural properties of both finite poles and the infinite pole of linear time-invariant singular systems under output feedback. Three main problems are studied, namely, (1) the algebraic structures of both finite poles and the infinite pole; (2) the assignability of finite poles and the elimination of the infinite pole by output feedback; and (3) the controllability and observability of the system with minimal number of inputs and outputs. New generic solutions to these problems are presented in terms of some new concepts defined in this paper including the geometric multiplicity of the infinite pole, the finite and impulsive output feedback cycle indices of the system. Determination of these multiplicities and indices are discussed. An assignability equivalence is established between the variable finite poles and the poles of a controllable and observable non-singular system. The number of the independent infinite poles that can be reduced is given in terms of the system matrices. The minimal number of inputs and outputs that guarantee controllability and observability are shown to be the output feedback cycle indices.

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1. Introduction

In the past two decades, a considerable amount of research concerning linear time-invariant (LTI) singular systems (or descriptor systems) has been reported because of their extensive applications. It is well known that a singular system has complicated structures and contains not only finite poles but also infinite poles which may generate undesired impulsive behaviors (Verghese, Levy, & Kailath, 1981). Efforts have been devoted to investigating structural properties of this kind of singular systems. Results on controllability and observability, stability, pole assignment, and feedback regularization have been established by both algebraic and geometric approaches (see, for example, Dai, 1989; Lewis, 1986 and the references therein). Cobb (1981) shows that the finite poles can be freely assigned and the infinite poles can all be eliminated by state feedback if the system is strongly controllable (i.e., both R-controllable and impulse controllable). In the case of output feedback, the theoretic results are available mainly for the elimination of the infinite pole, less is known about the assignment of the finite poles. Under the assumption of impulse controllability and observability, the closed-loop system can be made impulse free by output feedback (Dai, 1989; Zhang, 1989). Wang, Lin, and Xie (1994) calculate the number of the infinite poles that are eliminable based-on the slow/fast decomposition of the system. Topics related to decentralized output feedback control have also been studied by several researchers (see, for example, Chang & Davison, 2001; Wang & Soh, 1999).

This paper studies three basic problems regarding the algebraic structural properties of singular systems. They are, specifically, (1) the algebraic structures of both finite poles and the infinite pole; (2) the assignability of finite poles and the elimination of the infinite pole by output feedback; and (3) the controllability and observability of the system with
minimal number of inputs and outputs. New generic solutions to these problems are presented. First, the concepts regarding the finite pole structure and the infinite pole structure of the open-loop singular system are introduced. New concepts of the finite and impulsive output feedback cycle indices that characterize the ability of output feedback to reducing the multiplicities and indices of the open-loop system are defined. Determination of these multiplicities and indices are discussed. It is shown that under almost any output feedback all the closed-loop finite poles, except the fixed ones (corresponding to the uncontrollable or unobservable modes), can be assigned to be distinct and shifted away from any given finite set. The number of the infinite poles that are eliminable is given in terms of these multiplicities. The minimal number of inputs and outputs that guarantee controllability and observability are shown to be the output feedback cycle indices defined in this paper.

The remaining part of the paper is organized as follows. The concepts regarding finite and infinite pole structures are presented in Sections 2 and 3 for open- and closed-loop singular systems, respectively. Elimination of the infinite poles is studied in Section 4. Section 5 establishes the so-called assignability equivalence between the variable finite poles of the closed-loop system and the poles of a non-singular system. The determination of the multiplicities and indices is discussed in Section 6. Section 7 presents generic characterization of controllability and observability of the singular system using cycle indices. Section 8 gives the conclusion.

Some proofs in this paper are straightforward, and are omitted.

Notations: The notation of this paper is standard. \( \Phi \) denotes the empty set. For a linear space \( \mathcal{X} \), \( \dim(\mathcal{X}) \) is the dimension of \( \mathcal{X} \). For a matrix \( M \), \( \text{col}(M) \) and \( \text{row}(M) \) denote the number of columns and the number of rows of \( M \), respectively, and \( \text{null}(M) \) the null space of \( M \).

### 2. Open-loop pole structure and system indices

Consider a LTI singular system \( \Sigma \):

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx,
\end{align*}
\]  

(1)

where \( x \in \mathbb{R}^n \) is the state of the system, \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^m \) are the input and output vectors of the system respectively, \( E \in \mathbb{R}^{m \times n} \) is assumed to be singular with \( 0 < \text{rank}(E) = q < n \) and \( A, B, C \) are real constant matrices of appropriate sizes. For brevity, we also denote system (1) by \( \Sigma = (E, A, B, C) \).

We recall some basic definitions. System (1) is said to be regular if \( \det(sE - A) \neq 0 \). If \( \deg(\det(sE - A)) = n_1 \), then system (1) has \( n_1 \) finite poles (counted repeatedly for multiple poles), defined as the eigenvalues of the matrix pair \((E, A)\). Denote by \( \sigma(E, A) \) the set of these finite eigenvalues. For any finite pole \( \lambda \in \sigma(E, A) \), the number \( \text{gm}(\lambda, E, A) = \dim[\text{null}(\lambda E - A)] \) is defined as the geometric multiplicity of \( \lambda \).

**Definition 1.** The finite cycle index (FCI) of system (1) is defined as

\[
\text{cyc}(E, A) = \max\{\dim[\text{null}(\lambda E - A)], \lambda \in \mathbb{C}\}.
\]

Note that if \( \sigma(E, A) \neq \emptyset \), then \( \text{cyc}(E, A) = \max\{\text{gm}(\lambda, E, A), \lambda \in \sigma(E, A)\} \) since \( \det(\lambda E - A) = 0 \) for \( \lambda \not\in \sigma(E, A) \). System (1) is regular but has no finite pole if and only if \( \text{cyc}(E, A) = 0 \). And if \( (E, A) \) has only distinct finite poles, then \( \text{cyc}(E, A) = 1 \).

System (1) may have a pole at infinity, the so-called infinite pole (or impulsive mode), if \( \deg(\det(sE - A)) < \text{rank}(E) \). The degree deficiency of \( \det(sE - A) \) is defined (Verghese et al., 1981) as its algebraic multiplicity of the infinite pole, and denoted by \( \text{alg}_{\infty}(E, A) \). i.e.,

\[
\text{alg}_{\infty}(E, A) = \text{rank}(E) - \deg(\det(sE - A)).
\]

In case that \( \det(sE - A) = 0 \), \( \deg(\det(sE - A)) = -\infty \) by convention, thus \( \text{alg}_{\infty}(E, A) = \infty \).

We now consider defining geometric multiplicity for the infinite pole. Let the Smith canonical form of \( E - \lambda A \) be \( \text{diag}(\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_p(\lambda), 0) \), where \( \mu \) is the normal rank of \( E - \lambda A \), and \( \varphi_1(\lambda), \ldots, \varphi_p(\lambda) \) are polynomials of \( \lambda \) with leading coefficient one, satisfying \( \varphi_1(\lambda) | \varphi_2(\lambda) | \cdots | \varphi_p(\lambda) \), and \( O \) stands for the zero block matrix of appropriate size. The infinite rank of \( sE - A \) is first defined by Xie, Wang, Lin, and Cheng (1995) as \( \text{rank}_{\infty}(sE - A) = \max\{k | \lim_{\lambda \to 0} \varphi_k(\lambda)/\lambda \neq 0 \text{, and } \lim_{\lambda \to 0} \varphi_{k+1}(\lambda)/\lambda = 0 \} \). It is given in Wang, Soh, and Xie (1997) that

\[
\text{rank}_{\infty}(sE - A) = \text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} - \text{rank}(E). \tag{2}
\]

**Definition 2.** The geometric multiplicity (GM) of the infinite pole of system (1) is defined as

\[
\text{gm}_{\infty}(E, A) = n - \text{rank}_{\infty}(sE - A).
\]

The geometric multiplicity of the infinite pole is also called the impulsive cycle index (ICI) of system (1), and denoted by \( \text{cyc}_{\infty}(E, A) \). The overall cycle index (OCI) of system (1) is defined as

\[
\text{Cyc}(E, A) = \max\{\text{cyc}(E, A), \text{cyc}_{\infty}(E, A)\}.
\]

Note that the definition of the geometric multiplicity given by Dai (1989) is in terms of the fast subsystem of a restricted equivalence decomposition of system (1), which is applicable only to regular systems.

The algebraic multiplicity of the infinite pole is also called the number of the impulsive modes, and the geometric multiplicity is referred to as the number of the independent infinite poles. They are also called, respectively, the impulsive algebraic multiplicity and the impulsive geometric multiplicity of system (1), which sounds more logical if it has no infinite pole.
Remark 1. It can be easily verified that the finite cycle index, the impulsive cycle index, and the overall cycle index of system (1) are invariant under restricted equivalence transformation (Gantmacher, 1959).

If system (1) is regular, it is restrictively equivalent to a standard slow/fast subsystem decomposition form (see, e.g., Dai, 1989). That is, there is a state transformation \( \hat{x} = P_1^{-1} \) and a non-singular matrix \( Q_1 \) so that system (1) becomes
\[
\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad \hat{y} = \hat{C}\hat{x},
\]
where
\[
\hat{E} = Q_1EP_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix},
\]
\[
\hat{A} = Q_1AP_1 = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},
\]
\[
\hat{B} = Q_1B = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C} = CP_1 = [\hat{C}_1 \hat{C}_2]
\]
with \( n_2 = n - n_1 \) and \( N \in \mathbb{R}^{n_1 \times n_2} \) being nilpotent.

Direct calculations result in the following
\[
a\lg_{\infty}(E,A) = \text{rank}(N)
\]
and
\[
\text{gm}_{\infty}(E,A) = \text{rank}(N) - \text{rank}(N^2) \leq a\lg_{\infty}(E,A).
\]
Expression (6) also gives the difference between the algebraic and geometric multiplicities of the infinite pole. Since \( N \) is nilpotent, the two multiplicities are identical if and only if both of them are either 0 or 1.

3. Closed-loop pole structure and system indices

Applying the static output feedback
\[
u = Ky + v,
\]
to system (1), we have the closed-loop system
\[
E\dot{x} = (A + BKC)x + Bv, \quad y = Cx.
\]
The fixed polynomial (FP) of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) can be defined as
\[
p_f(s, \Sigma) = \gcd\{\det(sE - A - BKC), K \in \mathbb{R}^{r \times m}\},
\]
where \( \gcd \) stands for greatest common divisor and \( \det(sE - A - BKC) \) is assumed not to be identically zero for some \( K \in \mathbb{R}^{r \times m} \). The zeros of the fixed polynomial are the finite fixed modes (FFM) of system (1) that is given by \( A(\Sigma) = \bigcap_{K \in \mathbb{R}^{r \times m}} \sigma(E,A + BKC) \subset \sigma(E,A) \). System (1) is said to have impulsive fixed mode (IFM) if the closed-loop system (8) has infinite pole for any \( K \in \mathbb{R}^{r \times m} \).

Definition 3. The output feedback variable polynomial (VP) of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) is defined as
\[
p_{\alpha}(s, \Sigma, K) = \frac{\det(sE - A - BKC)}{p_f(s, \Sigma)}.
\]
Since the algorithms for determination of FFM are available (Dai, 1989), the variable polynomial can then be determined accordingly. For convenience, we also use the notation \( p_f(s) \) and \( p_f(s,K) \) to denote, respectively, the fixed polynomial and the variable polynomial.

Remark 2. The zeros of the variable polynomial are the finite poles of the closed-loop system (8) that are variable as the output feedback law changes. They are, therefore, called the variable finite poles of system (1). It follows from the structural decomposition (see, e.g., Dai, 1989) of system (1) that the FFM are those modes that are either not strongly controllable, or not strongly observable. In other words, \( p_{\alpha}(s,K) = \det(sE_{co} - A_{co} - B_{co}KC_{co}) \) where \( (E_{co}, A_{co}, B_{co}, C_{co}) \) is the strongly controllable and observable subsystem of system (1).

Definition 4. The geometric multiplicity of an FFM \( \lambda \in \lambda(\Sigma) \) is defined as
\[
\text{gm}(\lambda, \Sigma) = \min\{\text{gm}(E,A + BKC), K \in \mathbb{R}^{r \times m}\}.
\]
The finite output feedback cycle index (FOCI) of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) is defined as
\[
\text{cyc}(\Sigma) = \min\{\text{cyc}(E,A + BKC), K \in \mathbb{R}^{r \times m}\}.
\]
To characterize the ability of the output feedback to change the infinite pole structure of system (1), we introduce the following definitions.

Definition 5. The algebraic multiplicity and the geometric multiplicity of the IFM of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) is defined, respectively, as
\[
a\lg_{\infty}(\Sigma) = \min\{a\lg_{\infty}(E,A + BKC), K \in \mathbb{R}^{r \times m}\},
\]
and
\[
\text{gm}_{\infty}(\Sigma) = \min\{\text{gm}_{\infty}(E,A + BKC), K \in \mathbb{R}^{r \times m}\}.
\]
The algebraic and the geometric multiplicities of the IFM of the system may also be referred to as the impulsive output feedback algebraic multiplicity and the impulsive output feedback geometric multiplicity of the system, respectively.

Definition 6. The impulsive output feedback cycle index (IOCI) of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) is defined as
\[
\text{cyc}_{\infty}(\Sigma) = \min\{\text{cyc}_{\infty}(E,A + BKC), K \in \mathbb{R}^{r \times m}\}.
\]
And the overall output feedback cycle index (OOCI) of system (1) (w.r.t. \( \mathbb{R}^{r \times m} \)) is defined as
\[
\text{Cyc}(\Sigma) = \min\{\text{Cyc}(E,A + BKC), K \in \mathbb{R}^{r \times m}\}.
\]
Proposition 1. The algebraic and geometric multiplicities and the cycle indices defined in Definitions 4–6 are invariant under restricted equivalence transformation and/or output feedback.

Remark 3. Note that the multiplicities and the cycle indices defined in Definitions 4–6 are in fact the smallest respective multiplicities or indices of the closed-loop system (8) when the output feedback law varies. Thus, they are not greater than the respective multiplicities or indices of the open-loop system (1). Moreover, it follows from Lemma A in Appendix that almost any output feedback matrix $K \in \mathbb{R}^{r \times m}$ will yield the respective number. Accordingly,

$$0 \leq \text{cyc}_\infty(\Sigma) = \text{gm}_\infty(\Sigma) \leq \text{alg}_\infty(\Sigma)$$

and for almost any $K \in \mathbb{R}^{r \times m}$

$$\text{deg}(p_n(s, K)) = \text{rank}(E) - \text{alg}_\infty(\Sigma)$$

which is equal to the number of finite poles of the closed-loop system (8).

4. Elimination of infinite poles

Since the total number of finite and infinite poles of system (1) is constant, we may increase the number of finite poles by reducing the number of infinite poles. Equivalently, some infinite poles may be shifted to finite ones.

Theorem 1. The number of the infinite poles (counted repeatedly) of system (1) that can be eliminated by output feedback (7) is given by

$$a_\infty = \text{alg}_\infty(E, A) - \text{alg}_\infty(\Sigma).$$

Furthermore, the elimination can be achieved by almost any $K \in \mathbb{R}^{r \times m}$.

Theorem 2. The number of the independent infinite poles of system (1) that can be reduced by output feedback (7) is given by

$$g_\infty = \text{gm}_\infty(E, A) - \text{gm}_\infty(\Sigma)$$

$$= \min \left\{ \text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix}, \text{rank} \begin{bmatrix} E & A \\ 0 & E \end{bmatrix} \right\} - \text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix}.$$ (9)

Moreover, the elimination can be achieved by almost any $K \in \mathbb{R}^{r \times m}$.

Eq. (10) follows from Definitions 2 and 5, Eq. (2) and Lemma B in Appendix.

Remark 4. It follows that the infinite poles of system (1) can all be eliminated by almost any output feedback (7) if and only if $\text{alg}_\infty(\Sigma) = 0$ (or $\text{gm}_\infty(\Sigma) = 0$, or $\text{cyc}_\infty(\Sigma) = 0$). Also, a direct application of Theorem 2 leads to the following well-known result (e.g., Zhang, 1989) that the infinite poles can all be eliminated by almost any output feedback (7) if and only if system (1) is both impulse controllable and observable.

Theorems 1 and 2 show that both the impulse algebraic multiplicity and the impulse geometric multiplicity tell how far a singular system is away from impulse free, and that the multiplicities of the IFM indicate how much the singular system can be made close to impulse free by output feedback.

5. Assignment of finite poles

An assignability equivalence between the variable finite poles of system (1) and the poles of a controllable and observable non-singular system is established in this section.

Let $Q_2, P_2$ be non-singular matrices satisfying $Q_2EP_2 = \text{diag}(I_q, 0)$. System (1) is restricted equivalent to

$$\tilde{E}\ddot{x} = \tilde{A}\dot{x} + \tilde{B}u, \quad y = \tilde{C}x, \quad (11)$$

where the new state $\tilde{x} = P_2^{-1}x$, and

$$\tilde{E} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = Q_2AP_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$\tilde{B} = Q_2B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \tilde{C} = CP_2 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$ (12)

If system (1) is impulse controllable and observable, then $\text{rank}[A_{22} B_2] = n - q$ and $\text{rank}\begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} = n - q$. From Lemma B in Appendix it follows that $A_{22} + B_2\tilde{C}_2$ is invertible for almost any $\tilde{K} \in \mathbb{R}^{r \times m}$. For such a matrix $\tilde{K}$, letting the output feedback be $u = \tilde{K}y + v$, System (11) then has the following closed-loop form

$$\tilde{E}\ddot{x} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \ddot{x} + \tilde{B}v, \quad y = \tilde{C}x, \quad (13)$$

where $X_{ij} = A_{ij} + B_1\tilde{C}_j, i, j = 1, 2$. Define two non-singular matrices

$$Q_3 = \begin{bmatrix} I_q & -X_{12}X_{22}^{-1} \\ 0 & I_{n-q} \end{bmatrix}, \quad P_3 = \begin{bmatrix} I_q & 0 \\ -X_{22}^{-1}X_{21} & X_{22}^{-1} \end{bmatrix}.$$ (14)

Under the state transformation $\tilde{x} = P_3^{-1}\ddot{x}$, system (12) is restricted equivalent to

$$\ddot{\tilde{x}} = \tilde{A}\dot{\tilde{x}} + \tilde{B}v, \quad y = \tilde{C}x, \quad (15)$$
where
\[
\hat{E} = \tilde{E}, \quad \hat{A} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-q} \end{bmatrix},
\]
(14)
\[
\hat{B} = \begin{bmatrix} B_s \\ B_t \end{bmatrix}, \quad \hat{C} = [ C_s \quad C_t ]
\]
with \( A_s = X_{11} - X_{12}X_{22}^{-1}X_{21}, B_s = B_1 - X_{12}X_{22}^{-1}B_2, B_t = B_2, \\
C_s = C_1 - C_2X_{22}^{-1}X_{21}, C_t = C_2X_{22}^{-1} \).

Noting also that R-controllability and R-observability are invariant under restricted equivalence transformation and output feedback, we can obtain the following decomposition of the closed-loop system.

**Lemma 1.** If system (1) is impulse controllable and observable, then for almost any output feedback (7), the closed-loop system (8) is restricted equivalent to a system with the decomposition given in (13) and (14). If system (1) is also R-controllable and observable, then the subsystem \((A_s, B_s, C_s)\) of order \(q\) is completely controllable and observable.

**Theorem 3.** Assume that system (1) is strongly controllable and observable. The assignability of its finite poles by output feedback is equivalent to that of the poles of a completely controllable and observable non-singular system \((A_s, B_s, C_s)\) of order \(q\).

**Proof.** Applying the output feedback \(v = \tilde{K}y + w\) to system (13), we get the characteristic equation of the resultant closed-loop system
\[
d(s) = \det \begin{bmatrix} sI - A_s - B_s\tilde{K}C_s & -B_s\tilde{K}C_t \\
-B_t\tilde{K}C_s & -I - B_t\tilde{K}C_t \end{bmatrix} = 0.
\]
Noting that \(I + B_t\tilde{K}C_t\) is invertible for almost any \(\tilde{K} \in \mathbb{R}^{r \times m}\), we have
\[
d(s) = \det(-I - B_t\tilde{K}C_t) \det(sI - A_s \\
- B_s\tilde{K}C_s + B_s\tilde{K}C_t(I + B_t\tilde{K}C_t)^{-1}B_t\tilde{K}C_s \\
- B_t\tilde{K}C_s + B_t\tilde{K}C_t) = 0.
\]
Also, \(I - K_sC_sB_t\) is invertible for almost any \(K_s \in \mathbb{R}^{r \times m}\). Under the invertibility constraints, it can be verified by direct matrix manipulations using the matrix identities
\[
Y(I + XY)^{-1} = (I + XY)^{-1}Y \text{ and } (I + XY)^{-1} = I - (I + X)^{-1}
\]
that \(K_s = \tilde{K} - \tilde{K}C_t(I + B_t\tilde{K}C_t)^{-1}B_t\tilde{K}\) if and only if \(K_s = (I - K_sC_sB_t)^{-1}K_s\). Therefore, the set of the finite poles
\[
\sigma(A_s + B_sK_sC_s) = \sigma(\tilde{E}, \tilde{A} + B\tilde{K}C) = \sigma(\tilde{E}, \tilde{A} + B\tilde{K}\tilde{C} + B\tilde{K}\tilde{C})
\]
where \(K = \tilde{K} + \tilde{K}\). The assignability equivalence then follows from Remark A in Appendix and the generic property of the matrices \(\tilde{K}\) and \(K_s\) that yield the same desired poles. This completes the proof. \(\blacksquare\)

The above theorem implies that under the assumption of strong controllability and observability what we can say about the assignment of the variable finite poles of the closed-loop singular system is as much as what we understand about the assignment of the poles of a controllable and observable triple \((A_s, B_s, C_s)\) of order rank(\(E\)). We call this the equivalence of pole assignability under output feedback.

Using the notion of the variable polynomial, this equivalence can be extended to any singular systems that are not necessarily strongly controllable and observable. Consequently, all the results on output feedback for the non-singular system can be extended straightforward to the singular system. The general equivalence, together with the extension of a result by Davison and Wang (1972) about the separation and shifting of the finite poles, is presented in the next theorem.

**Theorem 4.** The assignability of the variable finite poles of system (1) is equivalent to that of the poles of a controllable and observable triple of order rank(\(E\)) \(-\text{alg}_\infty(\Sigma)\). Consequently, for almost any output feedback (7) these finite poles are distinct and disjoint with any given finite set in the complex plane.

**Remark 5.** If system (1) is strongly controllable and observable, then almost any output feedback (7) can eliminate the infinite poles and in the same time assign the closed-loop finite poles distinct and disjoint with any given finite set in the complex plane.

### 6. Determination of multiplicities and indices

The algebraic and geometric multiplicities of any open-loop finite pole and the infinite pole can be determined directly by their definitions. As for the indices of the closed-loop system, it follows from Lemma A in Appendix that it is generically true that any output feedback matrix \(K \in \mathbb{R}^{r \times m}\) will result in the respective index. Specifically, for almost any \(K \in \mathbb{R}^{r \times m}\), we have
\[
gm(\lambda, \Sigma) = g_m(\lambda, E, A + BKC),
\]
\[
\nu(\Sigma) = \nu(E, A + BKC),
\]
where \(\nu\) may represent any of the indices \(\text{alg}_{\infty}, \text{gm}_{\infty}, \text{cyc}, \text{cyc}_{\infty}, \text{and Cyc}\). This means that any index can be determined from the pair \((E, A + BKC)\) with almost any randomly selected \(K \in \mathbb{R}^{r \times m}\). In particular, if two such matrices yield the same number, we can be practically sure that the number is the relevant index indeed. In addition, due to the generic
property of the indices involved, we get

\[ \text{Cyc}(\Sigma) = \max \{ \text{cyc}(\Sigma), \text{cyc}_\infty(\Sigma) \}. \]

The next three theorems present simple formulas to determine the geometric multiplicity of an FFM, the output feedback cycle indices of system (1) in terms of the system matrices explicitly.

**Theorem 5.** Let \( \lambda \) be an FFM of system (1). Its geometric multiplicity can be calculated by

\[ \text{gm}(\lambda, \Sigma) = n - \min \left\{ \text{rank}[\lambda E - A B], \text{rank} \left[ \begin{array}{c} \lambda E - A \\ C \end{array} \right] \right\}. \]

We now turn to the finite output feedback cycle index of system (1). Let us first examine an extreme case where the closed-loop system (8) always has no finite pole, i.e., for any \( K \in \mathbb{R}^{r \times m} \), \( \det(sE - A - BKC) \) is equal to some non-zero constant. Accordingly, \( \text{cyc}(\Sigma) = 0 \). Except this special case, the finite output feedback cycle index of system (1) can be determined in terms of its FFM as given in the next theorem.

**Theorem 6.** Except the special case mentioned above, the finite output feedback cycle index of system (1) can be determined as

\[ \text{cyc}(\Sigma) = \begin{cases} \max \{ \text{gm}(\lambda, \Sigma), \lambda \in A(\Sigma) \} & \text{if } A(\Sigma) \neq \Phi, \\ 1, & \text{otherwise}. \end{cases} \]

**Proof.** If \( A(\Sigma) \neq \Phi \), then for any given \( K \in \mathbb{R}^{r \times m} \), \( A(\Sigma) \subset \sigma(E, A + BKC) \neq \Phi \), and

\[ \text{cyc}(E, A + BKC) = \max \{ \text{dim}(\text{null}(\lambda E - A - BKC)), \lambda \in \sigma(E, A + BKC) \} \]

\[ \geq \max \{ \text{dim}(\text{null}(\lambda E - A - BKC)), \lambda \in A(\Sigma) \} \]

\[ \geq \max \{ \text{gm}(\lambda, \Sigma), \lambda \in A(\Sigma) \}. \]

From Theorems 4, we know that for almost all \( K \in \mathbb{R}^{r \times m} \), \( p_v(s, K) \) has only distinct zeros that are not in \( A(\Sigma) \). Noting the generic property given in (15) and the fact that the intersection of any finite set of Zariski open sets is still a Zariski open set, we have that for almost all \( K \in \mathbb{R}^{r \times m} \), \( \text{cyc}(E, A + BKC) = \max \{ \text{gm}(\lambda, \Sigma), \lambda \in A(\Sigma) \} \). The result then follows from Definition 4.

In the case of \( A(\Sigma) = \Phi \), \( \det(sE - A - BKC) = p_v(s, K) \). Hence for almost all \( K \in \mathbb{R}^{r \times m} \), \( (E, A + BKC) \) has only distinct finite eigenvalues. Thus, \( \text{cyc}(\Sigma) = 1 \). This completes the proof. \( \square \)

The main contribution of Theorem 6 is the characterization of the relationship between the geometric multiplicities of FFM and the finite output feedback cycle index of the system. In order to determine the latter, it is more convenient to compute \( \text{cyc}(E, A + BKC) \) directly for an arbitrary \( K \in \mathbb{R}^{r \times m} \) without a priori knowledge on FFM. Furthermore, if \( \text{cyc}(\Sigma) \) is found to be greater than one, then system (1) must have FFM.

**Theorem 7.** The impulsive output feedback cycle index of system (1) can be calculated by

\[ \text{cyc}_\infty(\Sigma) = n + \text{rank}(E) \]

\[ - \min \left\{ \text{rank} \left[ \begin{array}{cc} E & 0 \\ A & E \end{array} \right], \text{rank} \left[ \begin{array}{cc} E & A \\ 0 & E \end{array} \right] \right\}. \]

**7. Controllability and observability**

This section considers how the cycle indices of system (1) characterize its controllability and/or observability.

**Proposition 2.** Assume that system (1) is regular, and denote \( \Sigma_F = (E, A, F, C) \), Then

(a) if \( \Sigma \) is R-controllable, then \( \text{rank}(B) \geq \text{cyc}(E, A) \)

Moreover, for almost any \( F \in \mathbb{R}^{r \times \text{cyc}(E, A)} \), \( \Sigma_F \) is R-controllable;

(b) if \( \Sigma \) is impulse controllable, then \( \text{rank}(B) \geq \text{cyc}_\infty(E, A) \)

Moreover, for almost any \( F \in \mathbb{R}^{r \times \text{cyc}_\infty(E, A)} \), \( \Sigma_F \) is impulse controllable;

(c) if \( \Sigma \) is strongly controllable, then \( \text{rank}(B) \geq \text{cyc}(E, A) \)

Moreover, for almost any \( F \in \mathbb{R}^{r \times \text{cyc}(E, A)} \), \( \Sigma_F \) is strongly controllable.

**Proof.** (a) If system (1) is regular, it has the standard decomposition (3–5). Then it is R-controllable if and only if \( \text{rank}[I_{n_1} - \hat{A}_1 \hat{B}_1] = n_1 \) for any \( s \in \mathbb{C} \). From the property of the cycle index of matrix \( A_1 \), we obtain that \( \text{rank}(B) \geq \text{rank}(\hat{B}_1) \geq \text{cyc}(\hat{A}_1) = \text{cyc}(E, A) \).

Also from the property of the cycle index of matrix \( \hat{A}_1 \), it follows that there exists an \( F \in \mathbb{R}^{r \times \text{cyc}(E, A)} \) such that \( \Sigma_F \) is R-controllable. The genericity of the controllability follows directly from the generic property of matrix rank.

(b) System (1) is impulse controllable if and only if

\[ \text{rank} \left[ \begin{array}{cc} N & 0 \\ I_{n_2} & N \hat{B}_2 \end{array} \right] = n_2 + \text{rank}(N), \text{ i.e.,} \]

\[ \text{rank} \left[ N^2 \hat{B}_2 \right] = \text{rank}(N). \]

This implies that \( \text{rank}(N \hat{B}_2) \geq \text{rank}(N) - \text{rank}(N^2) = \text{cyc}_\infty(E, A) \). Therefore \( \text{rank}(B) \geq \text{cyc}_\infty(E, A) \).

Assume, without loss of generality, that \( N = \text{diag}[O, J] \) where \( O \in \mathbb{R}^{n_0 \times n_0} \) stands for the zero matrix and \( J \in \mathbb{R}^{(n_2 - n_0) \times (n_2 - n_0)} \) represents the standard non-zero Jordan blocks of \( N \). It can be shown that there exists a matrix \( \hat{B}_2 \in \mathbb{R}^{(n_2 - n_0) \times \text{cyc}_\infty(E, A)} \) such that the matrix \([J \hat{B}_2]\) is of full
row rank. Let \( \tilde{B}_2 = [\tilde{b}_1 \tilde{b}_2] \) with \( \tilde{b}_2 \) being the zero matrix of \( n_0 \times \text{cyc}_\infty(E, A) \), Eq. (16) is satisfied. Hence, there exists \( F \in \mathbb{R}^{p \times \text{cyc}_\infty(E, A)} \) such that \( \Sigma_F \) is impulse controllable. The genericity is obvious.

Part (c) is implied by parts (a) and (b).

From the principle of duality, Proposition 2 immediately leads to the following proposition on observability.

**Proposition 3.** Assume that system (1) is regular. Then if it is strongly (R-, or impulse) observable, then \( \text{rank}(C) \geq \text{Cyc}(E, A) \) (\( \text{cyc}(E, A) \), or \( \text{cyc}_\infty(E, A) \)). Moreover, for almost any \( G \in \mathbb{R}^{\text{Cyc}(E, A) \times n} \) (\( \mathbb{R}^{\text{cyc}(E, A) \times n} \), or \( \mathbb{R}^{\text{cyc}_\infty(E, A) \times n} \)), \( (E, A, B, G) \) is strongly (R-, or impulse) observable.

**Remark 6.** Propositions (2) and (3) imply that the minimal numbers of the inputs that lead to strong (R-, or impulse) controllability is the same as the minimal number of the outputs required to have the system strongly (R-, or impulse) observable. And the number is equal to the overall (finite, or impulse) cycle index of the system.

Propositions 2 and 3 together lead to the following characterizations of controllability and observability of the closed-loop system (8) via external inputs and outputs.

**Theorem 8.** Denote \( \Sigma_K = (E, A + BKC, F, G) \), the following statements are true.

(a) Let \( k = \text{cyc}(\Sigma) \). If \( \Sigma_K \) is R-controllable (R-observable) for some \( K \in \mathbb{R}^{r \times m} \), then \( \text{col}(F) \geq k \) (row \( (G) \geq k \)). Moreover, for almost any \( K \in \mathbb{R}^{r \times m} \) and \( F \in \mathbb{R}^{n \times k} \) \( (G \in \mathbb{R}^{k \times n}) \), \( \Sigma_K \) is R-controllable (R-observable).

(b) Let \( \text{cyc}_\infty(\Sigma) = k \). If \( \Sigma_K \) is impulse controllable (observable) for some \( K \in \mathbb{R}^{r \times m} \), then \( \text{col}(F) \geq k \) (row \( (G) \geq k \)). Moreover, for almost any \( K \in \mathbb{R}^{r \times m} \) and \( F \in \mathbb{R}^{n \times k} \) \( (G \in \mathbb{R}^{k \times n}) \), \( \Sigma_K \) is impulse controllable (observable).

(c) Let \( \text{Cyc}(\Sigma) = k \). If \( \Sigma_K \) is strongly impulse controllable (observable) for some \( K \in \mathbb{R}^{r \times m} \), then \( \text{col}(F) \geq k \) (row \( (G) \geq k \)). Moreover, for almost any \( K \in \mathbb{R}^{r \times m} \) and \( F \in \mathbb{R}^{n \times k} \) \( (G \in \mathbb{R}^{k \times n}) \), \( \Sigma_K \) is strongly impulse controllable (observable).

**Corollary 1.** If system (1) is strongly (R-, or impulse) controllable and observable, then for almost any \( K \in \mathbb{R}^{r \times m} \), \( f \) and \( g \in \mathbb{R}^{n} \), the system \( (E, A + BKC, f, g^T) \) is strongly (R-, or impulse) controllable and observable.

8. Conclusion

This paper studies the structural properties of both finite poles and the infinite pole of LTI singular systems under output feedback. The ability of output feedback to change the structure of finite and infinite pole of the system is characterized through some new concepts regarding the multiplicities of the poles or the indices of the system. The determination of these multiplicities and indices are discussed. The number of the infinite poles that can be eliminated by output feedback is given. An assignability equivalence is established between the variable finite poles and the poles of a controllable and observable non-singular system. It is consequently concluded that all the finite poles, except the fixed ones, can be separated from each other and shifted away from any given finite set. It is also shown that the minimal number of inputs (outputs) required for controllability (observability) is equal to the output feedback cycle indices of the singular system. The cycle indices under state feedback can also be defined and studied. They are, however, included in a special case in which all the state variables are assumed available. There are open problems, one of which is to characterize the algebraic multiplicity of the IFM in terms of only the system matrices.

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Appendix

Generic property and generic rank

Let \( x \in \mathbb{R}^{n} \), and \( f(x) \) denote a polynomial in the ring of polynomials \( \mathbb{R}[x] \). A set \( \mathscr{S} \subset \mathbb{R}^{n} \) is called a Zarisky open set (Fulton, 1995) if there is a polynomial \( f(x) \in \mathbb{R}[x] \) not identically zero such that

\[
\mathscr{S} = \mathbb{R}^{n} \setminus \mathscr{N}(f), \quad \mathscr{N}(f) = \{x \mid f(x) = 0, x \in \mathbb{R}^{n}\}.
\]

According to Wonham (1985), a property holding for a Zarisky open set is said to be a generic property. It is equivalent to say that this property holds for almost all points in \( \mathbb{R}^{n} \). Note that the union and intersection of any finite number of Zarisky open sets are still Zarisky open sets.

The above definition of Zarisky open set can be generalized to cover the subset of matrices \( K \in \mathbb{R}^{r \times m} \) by defining an associated vector \( k \) whose components are all the entries of matrix \( K \).

**Remark A.** Given matrices \( M \in \mathbb{R}^{r \times m} \) and \( N \in \mathbb{R}^{r \times r} \), with \( N \) full of rank. If \( \mathscr{S} \subset \mathbb{R}^{r \times m} \) is a Zarisky open set, then the sets \( \mathscr{S}_M = \{K_M \mid K_M = M + K, K \in \mathscr{S}\} \) and \( \mathscr{S}_N = \{K_N \mid K_N = NK, K \in \mathscr{S}\} \) are also Zarisky open sets in \( \mathbb{R}^{r \times m} \).

**Lemma A.** Let \( E \in \mathbb{R}^{p \times l} \) and \( p(K) \in \mathbb{R}^{p \times l} \) a matrix whose entries are polynomials in \( \mathbb{R}[k] \). Then for almost
any $K \in \mathbb{R}^{r \times m}$,

(a) $\text{g.r.} \{p(K)\} = \text{rank}[p(K)];$

(b) $\text{g.r.}_\infty \{sE - p(K)\} = \text{rank}_\infty [sE - p(K)];$

(c) $\max \{\text{deg}[\text{det}(sE - p(K))], K \in \mathbb{R}^{r \times m}\}$

$$= \text{deg}[\text{det}(sE - p(K))]$$

where $\text{g.r.}$ and $\text{g.r.}_\infty$ stand for generic rank and generic infinite rank, respectively, i.e., the maximum corresponding ranks as $K$ varies in $\mathbb{R}^{r \times m}$.

**Proof.** (a) Let $\text{g.r.} \{p(K), K \in \mathbb{R}^{r \times m}\} = r^*$, and denote $\mathscr{S} = \{K | \text{rank}[p(K)] = r^*\}$. Consider the polynomial $f(k)$ defined as the sum of the squares of all possible minors of order $r^*$ of the matrix $p(K)$, where the vector $k$ is an associated vector of $K$. It follows that $\mathscr{S} = \mathbb{R}^{r \times m} - \mathcal{N}(f)$. Since $r^*$ is attainable, $f(k)$ is not identically zero. Part (a) is proved.

(b) The equality follows from (2) and the result in (a).

(c) The result is valid since the leading coefficient of $\text{det}(sE - p(K))$ is a polynomial in $\mathbb{R}[k]$.

**Lemma B** (Xie, 1985, Xie et al., 1995). Let $E \in \mathbb{R}^{p \times 1}$, $B \in \mathbb{R}^{p \times r}$, and $C \in \mathbb{R}^{m \times 1}$ be constant matrices. Then

(a) $\text{g.r.} \{E + BKC\} = \min \left\{\text{rank}[E B], \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} \right\}$,

(b) $\text{g.r.}_\infty \{sE + A + BKC\}$

$$= \min \left\{ \text{g.r.}_\infty \{sE + AB\}, \text{g.r.}_\infty \begin{bmatrix} sE + A \\ C \end{bmatrix} \right\}$$

**References**


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