policies, we have $J^{\pi}(z) \geq M - z$ if $z \geq z^*$. This establishes the lemma, because this lower bound can be achieved by the policy $\eta_0(z) = \sigma$. □

Let J: $R \rightarrow R$ be a real-valued function. Let T be the dynamic programming operator, defined as

$$\begin{aligned} T(J)(z) &= \min\{M - z, \min_{i \in I}\{\mathrm{E}[h(z,i) + J(f(z,i))]\}\} \\ &= \min\{M - z, \min_{i \in I}\{c_i + \mathrm{E}[J(\max(Y_i, z))]\}\} \end{aligned}$$

where Y_i is an independent sample from the distribution F_i.

We have the following result.

Lemma 4: Let J: $R \rightarrow R^+$ be a monotone nonincreasing function. Then, $T(J)$ is also monotone nonincreasing.

Proof: Let $z_1, z_2 \in R$ and $z_1 \geq z_2$. Clearly, $\max(Y_i, z_1) \geq \max(Y_i, z_2)$ for each $i \in I$. Thus,

$$J(\max(Y_i, z_1)) \leq J(\max(Y_i, z_2)).$$

In light of the above equation for T, we have

$$T(J)(z_1) \leq T(J)(z_2)$$

establishing that $T(J)$ is monotone nonincreasing.

Lemma 5: Under BCC assumptions $\hat{J}$ is monotone nonincreasing on R.

Proof: Compactness of I implies compactness of the control space $U = I \cup \{\sigma\}$. This compactness and the continuity assumptions of BCC imply that the dynamic programming problem under consideration satisfies the assumptions of the semicontinuous model of [1, Ch. 9], and hence it follows that the iteration $J^{n+1} = T(J^n)$, initialized with the initial condition $J^0(z) \equiv 0$, converges monotonically to the optimal cost $\hat{J}$ (see [1, Proposition 9.17 and Corollary 9.17.2]). From Lemma 4, since the initial condition is monotone nonincreasing, J^n is also monotone nonincreasing for all n, establishing that the limit $\hat{J}$ will also be monotone nonincreasing. □

We can now show the main result of the paper.

Theorem 1: Under BCC assumptions the sampling policy π^* is the optimal policy and the function J^* is the optimal cost function.

Proof: From Lemma 3 we have $\hat{J}(z) = J^*(z)$ for $z \geq z^*$. Let $z < z^*$. Then, by Lemma 5, the optimal cost is monotone nonincreasing, so

$$\hat{J}(z) \geq \hat{J}(z^*) = M - z^* = J^*(z).$$

On the other hand, since $\hat{J}$ is the minimum cost

$$\hat{J}(z) \leq J^*(z).$$

Hence $\hat{J}(z) = J^*(z)$ for $z < z^*$ and we have shown that $\hat{J} = J^*$.

According to Lemma 2, J^* is the cost function corresponding to policy π^*, therefore, π^* is the optimal policy. □

Remark 1—Vector Sampling: The above result extends to the case of vector sampling (or parallel sampling). Assume that at each stage instead of one sample, d $(d > 1)$ samples are taken (possibly in parallel). For each $\underline{i} = (i_1, \cdots, i_d) \in I^d$, let $C_{\underline{i}}$ represent the cost of taking d samples from choices $i_1, \cdots, i_d$ and let $W_{\underline{i}}$ represent the maximum of the samples taken. The distribution of $W_{\underline{i}}$ is given by $F_{W_{\underline{i}}} = \Pi_{j=1}^{d} F_{Y_{i_j}}$.

The sequential sampling problem discussed in this paper is equivalent to this problem, posed in terms of collecting samples $W_{\underline{i}}$ given the distribution of $C_{\underline{i}}$ and the derived distributions $F_{W_{\underline{i}}}$. As a result, a consequence of the above theorem is that the optimal sampling scheme is to always sample from the same combination of choices, i.e., the combination that produces the highest index (note that the d choices may not all be the same).

References

[1] D. Bertsekas and S. Shreve, *Stochastic Optimal Control: Discrete Time Case*, Mathematics In Science and Engineering Series, vol. 139. New York: Academic, 1978.

[2] D. A. Castañon and Z. Tang, "A counterexample to adaptive partitioned random search to global optimization," *IEEE Trans. Automat. Contr.*, vol. 41, p. 310, 1996.

[3] Y. Chow and H. Robbins, "A martingale systems theorem and applications," in *Proc. 4th Berkeley Symp. Mathematical Statistics and Probability*, 1961, vol. 1, pp. 93–104.

[4] J. Gittins, *Multi-Armed Bandit Allocation Indices*, 1st ed. New York: Wiley, 1989.

[5] D. B. Rosenfield, "Optimal strategies for selling an asset," *Management Sci.*, vol. 29, no. 9, pp. 1051–1058, 1983.

[6] Z. Tang, "Adaptive partitioned random search to global optimization," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2235–2244, Nov. 1994.

[7] S. Streltsov and P. Vakili, "A nonmyopic utility function for statistical global optimization," to be published.

[8] M. L. Weitzman, "Optimal search for the best alternative," *Econometrica*, vol. 47, no. 3, pp. 641–654, 1979.

On Regularizing Singular Systems by Decentralized Output Feedback

Dianhui Wang and C. B. Soh

***Abstract*— This paper considers linear time-invariant decentralized singular systems which are either nonregular or, if they are regular, they have impulsive modes. It derives algebraic necessary and sufficient conditions for making a singular system both regular and impulse-free by decentralized output feedback control laws and decentralized proportional-plus-derivative output feedback control laws.**

***Index Terms*—Decentralized control, regularization, singular systems, output feedback.**

I. Introduction

Consider the following decentralized singular systems:

$$\begin{aligned} E\dot{x} &= Ax + \sum_{i=1}^{N} B_i u_i \\ y_i &= C_i x, \qquad i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

where E and A are $n \times n$ real matrices with E singular, x and y_i are state vector and outputs vectors, respectively, B_i and C_i are $n \times m_i$ real matrices and $l_i \times n$ real matrices, respectively.

System (1) is said to be regular if the pencil pair $sE - A$ is regular, i.e., $\det(sE - A)$ is not identically zero. It is well known that regularity of singular systems guarantees the existence and uniqueness of the solutions [1], [2]. Almost all of the given results

Manuscript received March 20, 1996; revised January 1, 1997.

D. Wang is with the Institute of Intelligent Control, Dalian Maritime University, Dalian, Liaoning, 116026 China (e-mail: dhwang@dlmu.edu.cn).

C. B. Soh is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 Singapore.

Publisher Item Identifier S 0018-9286(99)00574-7.

for singular systems assume regularity; see, for example, [3]–[5] and the bibliographies in these references. However, this assumption is unnecessarily strong so that it limits the analysis of a number of practical physical systems [5], [6]. Then, researchers paid attention to regularizing singular systems using various feedbacks (see, e.g., [7]–[10]). It is important, of course, to establish conditions that ensure the regularity of singular systems under feedback and, furthermore, to develop numerical algorithms for constructing regular closed-loop systems with desired system characteristics [9], [11], [12].

From the viewpoint of practical applications, regular systems that have no impulsive modes are especially significant. So it is desirable to derive conditions that ensure the existence of one feedback gain matrix such that many resulting closed-loop systems are both regular and impulse-free. Early results on regularization of singular systems using proportional (P) state feedback and proportional-plus-derivative (P-D) state feedback are reported in [6], [8], and [11]. They have shown that many nonregular systems can be regularized by P and/or P-D state feedbacks. In [7] and [9], algebraic necessary and sufficient conditions are given that ensure that an output feedback can be selected for making the closed-loop system both regular and impulse-free. In [10], based on a reduced form of the singular system, algebraic sufficient conditions are derived for making a nonregular system both regular and strongly controllable and strongly observable. These results mentioned above are only for centralized systems.

The objective of this paper is to derive algebraic necessary and sufficient conditions for the existence of decentralized output feedback control laws, $u_i = K_i y_i + v_i, i = 1, 2, \cdots, N$, that will make the singular system (1) both regular and impulse-free. Algebraic necessary and sufficient conditions are also presented for the existence of P-D decentralized output feedback control laws, $u_i = -L_i \dot{y}_i + K_i y_i + v_i, i = 1, 2, \cdots, N$, that will make the closed-loop system satisfy

$$\max_{\substack{L_1,\cdots,L_N \\ K_1,\cdots,K_N}} \deg\left\{\det\left[s\hat{E} - \left(A + \sum_{i=1}^{N} B_i K_i C_i\right)\right]\right\} = \max_{L_1,\cdots,L_N} \operatorname{rank}[\hat{E}] \tag{2}$$

where $\deg\{\cdot\}$ and $\det[\cdot]$ denote degree of a polynomial $\{\cdot\}$ and determinant of a matrix $[\cdot]$, respectively, and $\hat{E} = E + \Sigma_{i=1}^{N} B_i L_i C_i$.

The remainder of this paper is organized as follows. Some notations used throughout the paper and the supporting results are given in Section II. The main results and remarks are reported in Section III, and conclusions are given in Section IV.

II. Notation and Preliminaries

In this section, we introduce the notation and some supporting results used in the paper. Let $R^{n\times m}$ $(C^{n\times m})$ denote the set of $n \times m$ real (complex) matrices. If $M = [m_{ij}]_{n\times m} \in R^{n\times m}$, then M^T denotes the transpose of M, $\operatorname{rank}[M]$ denotes the rank of M, and $I_m \in R^{m\times m}$ denotes the identity matrix. If $M \in R^{n\times m}$ is a parameterized matrix of K, then we use $\underset{K}{\text{g.r.}}[M]$ to stand for the generic rank of the matrix M, i.e., the maximum rank of the matrix M as K varies in a specific set. Let $\mathcal{N}$ denote the set $\{1, 2, \cdots, N\}$ and φ is a nonempty subset of $\mathcal{N}$ with elements $i_1, i_2, \cdots, i_s$ ordered such that $i_1 < i_2 < \cdots < i_s$. Then we define B_φ and C_φ such that

$$C_\varphi = \begin{bmatrix} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_{i_s} \end{bmatrix} \quad \text{and} \quad B_\varphi = [B_{i_1}, B_{i_2}, \cdots, B_{i_s}].$$

Moreover, $\mathcal{P}(\mathcal{N})$ is a power set of $\mathcal{N}$, which is the set of all the subsets of $\mathcal{N}$, $\mathcal{N} - \varphi = \{x: x \in \mathcal{N} \text{ and } x \notin \varphi\}$. Let

$$\mathcal{S}_\mathcal{C} = \{K: K = \text{block diag}[K_1, K_2, \cdots, K_N], \; K_i \in R^{m_i \times l_i}, i \in \mathcal{N}\}.$$

Lemma 2.1: The singular system $E\dot{x} = Ax$ is regular and impulse-free if and only if

$$\operatorname{rank}\begin{bmatrix} 0 & E \\ E & A \end{bmatrix} = n + \operatorname{rank}[E]. \tag{3}$$

Proof: The proof can be found in [13]. □

The following concepts on robust set and supporting algebraic results are important and will be used in the next section.

Definition 2.1 [14]: A subset of $R^{m\times p}$ (respectively, $C^{m\times p}$) is a *robust subset* (i.e., Zariski open set) of $R^{m\times p}$ (respectively, $C^{m\times p}$) if it is nonempty and its complement is the set of solutions in $R^{m\times p}$ (respectively, $C^{m\times p}$) to a finite set of polynomial equations. Such sets are open and dense in $R^{m\times p}$ (respectively, $C^{m\times p}$), and each robust subset of $C^{m\times p}$ contains a largest subset which is a robust subset of $R^{m\times p}$. The intersection of two robust subsets of $R^{m\times p}$ (respectively, $C^{m\times p}$) is also robust in $R^{m\times p}$ (respectively, $C^{m\times p}$). Any union of robust subsets of $R^{m\times p}$ (respectively, $C^{m\times p}$) is also robust in $R^{m\times p}$ (respectively, $C^{m\times p}$).

Lemma 2.2: Let $A_0 \in R^{m\times n}$, $B \in R^{m\times h}$, and $C \in R^{l\times n}$ be fixed real matrices, and $K \in R^{h\times l}$ be a variable matrix. Then

$$\underset{k}{\text{g.r.}}[A_0 + BKC] = \min\left\{\operatorname{rank}[A_0, B], \operatorname{rank}\begin{bmatrix} A_0 \\ C \end{bmatrix}\right\} \tag{4}$$

and furthermore, the set

$$\mathcal{S}_\mathcal{R} = \{K: \operatorname{rank}[A_0 + BKC] = \underset{K\in R^{h\times l}}{\text{g.r.}}[A_0 + BKC]\} \tag{5}$$

is a robust set, or equivalently, $\operatorname{rank}[A_0 + BKC]$ reaches its maximum value for almost all $K \in R^{h\times l}$.

Proof: The first part of the result can be found in [15]. We now proceed with the rest of the proof. Without loss of generality, we assume that A_0 is an $m \times m$ matrix and $\text{g.r.}[A_0 + BKC] = \operatorname{rank}[A_0, B]$. Obviously, from Definition 2.1 the statement is true for the case of $[A_0, B]$ being of full rank. Let now $\operatorname{rank}[A_0, B] = r < m$. Then we can choose nonsingular matrices U and V such that

$$UA_0V = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad UB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad CV = [C_1, C_2] \tag{6}$$

where A_1 is a square nonsingular matrix. By compressing the rows of B_2 and columns of C_2 to full row rank and full column rank matrices, respectively, we can choose nonsingular matrices P and Q such that

$$PU[A_0, B]\begin{bmatrix} V & 0 \\ 0 & I_h \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & 0 & B_{21} \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} U & 0 \\ 0 & I_l \end{bmatrix}\begin{bmatrix} A_0 \\ C \end{bmatrix} VQ = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_{21} & 0 \end{bmatrix}. \tag{8}$$

Note that $PUA_0VQ = UA_0V$ and $\operatorname{rank}[A_0, B] = r \le \operatorname{rank}[A_0^T, C^T]^T$, we get

$$\operatorname{rank}[A_0 + BKC] = \operatorname{rank}[PUA_0VQ + PUBKCVQ] \tag{9}$$

$$\operatorname{rank}\begin{bmatrix} A_1 & B_1 \\ 0 & B_{21} \end{bmatrix} = r, \qquad \operatorname{rank}\begin{bmatrix} A_1 & 0 \\ C_1 & C_{21}^* \end{bmatrix} = r \tag{10}$$

where B_{21} has row full rank and C_{21}^*, which is a submatrix of C_{21}, has column full rank. Let

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}_{r\times r}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_{21} \end{bmatrix}_{r\times h}, \quad \hat{C} = [C_1, C_{21}^*]_{l\times r}. \tag{11}$$

Then

$$\operatorname{rank}[\hat{A}+\hat{B}K\hat{C}] = \underset{K\in R^{h\times l}}{\text{g.r.}}[\hat{A}+\hat{B}K\hat{C}] \tag{12}$$

holds for almost all $K \in R^{h\times l}$ which completes the proof. □

Lemma 2.3: Let $A_0 \in R^{m\times n}$, $P_i \in R^{m\times h_i}$, and $Q_i \in R^{l_i\times n}$ be fixed real matrices, and $K_i \in R^{h_i\times l_i}$ be variable matrices, $i = 1, 2\cdots, N$. Then

$$\begin{aligned}
&\underset{K_1,\cdots,K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N}P_iK_iQ_i\right]\\
&\quad= \min\left\{\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N-1}P_iK_iQ_i, P_N\right],\right.\\
&\qquad\qquad\left.\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\begin{bmatrix}A_0+\sum_{i=1}^{N-1}P_iK_iQ_i\\ Q_N\end{bmatrix}\right\}.
\end{aligned}\tag{13}$$

Proof: Suppose that

$$\underset{K_1,\cdots,K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N}P_iK_iQ_i\right] = \operatorname{rank}\left[A_0+\sum_{i=1}^{N}P_iK_i^*Q_i\right]. \tag{14}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\operatorname{rank}\left[A_0+\sum_{i=1}^{N}P_iK_i^*Q_i\right]\\
&\quad\le \underset{K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N-1}P_iK_i^*Q_i+P_NK_NQ_N\right]\\
&\quad= \min\left\{\operatorname{rank}\left[A_0+\sum_{i=1}^{N-1}P_iK_i^*Q_i, P_N\right],\right.\\
&\qquad\qquad\left.\operatorname{rank}\begin{bmatrix}A_0+\sum_{i=1}^{N-1}P_iK_i^*Q_i\\ Q_N\end{bmatrix}\right\}\\
&\quad\le \min\left\{\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N-1}P_iK_iQ_i, P_N\right],\right.\\
&\qquad\qquad\left.\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\begin{bmatrix}A_0+\sum_{i=1}^{N-1}P_iK_iQ_i\\ Q_N\end{bmatrix}\right\}.
\end{aligned}\tag{15}$$

Note that for any fixed $\tilde{K}_1, \cdots, \tilde{K}_{N-1}$, by Lemma 2.2 we have

$$\begin{aligned}
&\underset{K_1,\cdots,K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N}P_iK_iQ_i\right]\\
&\quad\ge \underset{K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N-1}P_i\tilde{K}_iQ_i+P_NK_NQ_N\right]\\
&\quad= \min\left\{\operatorname{rank}\left[A_0+\sum_{i=1}^{N-1}P_i\tilde{K}_iQ_i, P_N\right],\right.\\
&\qquad\qquad\left.\operatorname{rank}\begin{bmatrix}A_0+\sum_{i=1}^{N-1}P_i\tilde{K}_iQ_i\\ Q_N\end{bmatrix}\right\}.
\end{aligned}\tag{16}$$

Therefore

$$\begin{aligned}
&\underset{K_1,\cdots,K_N}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N}P_iK_iQ_i\right]\\
&\quad\ge \min\left\{\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\left[A_0+\sum_{i=1}^{N-1}P_iK_iQ_i, P_N\right],\right.\\
&\qquad\qquad\left.\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\begin{bmatrix}A_0+\sum_{i=1}^{N-1}P_iK_iQ_i\\ Q_N\end{bmatrix}\right\}.
\end{aligned}\tag{17}$$

This completes the proof. □

III. Main Result

In this section, we are interested in the effects of applying various local feedback control laws to system (1). First, we apply the feedback control law of the form $u_i = K_iy_i + v_i, i = 1, 2, \cdots, N$, to system (1). The closed-loop system is given by

$$E\dot{x} = \left[A+\sum_{i=1}^{N}B_iK_iC_i\right]x + \sum_{i=1}^{N}B_iv_i. \tag{18}$$

The main result is now stated as follows.

Theorem 3.1: Given the singular system (1), there exist decentralized control laws of the form $u_i = K_iy_i + v_i, i = 1, 2, \cdots, N$, that yield a regular and impulse-free system (18) if and only if for all $\varphi \in \mathcal{P}(\mathcal{N})$

$$\operatorname{rank}\begin{bmatrix}0 & E & 0\\ E & A & B_\varphi\\ 0 & C_{\mathcal{N}-\varphi} & 0\end{bmatrix} \ge n + \operatorname{rank}[E]. \tag{19}$$

Proof: According to Lemma 2.1, it is only needed to show that the following equality holds:

$$\underset{K_1,\cdots,K_N}{\text{g.r.}}\begin{bmatrix}0 & E\\ E & A+\sum_{i=1}^{N}B_iK_iC_i\end{bmatrix} = n + \operatorname{rank}[E]. \tag{20}$$

By Lemma 2.3 the necessary and sufficient condition for (20) to be true is that the following two inequalities are true:

$$\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\begin{bmatrix}0 & E & 0\\ E & A+\sum_{i=1}^{N-1}B_iK_iC_i & B_N\end{bmatrix} \ge n + \operatorname{rank}[E] \tag{21}$$

and

$$\underset{K_1,\cdots,K_{N-1}}{\text{g.r.}}\begin{bmatrix}0 & E\\ E & A+\sum_{i=1}^{N-1}B_iK_iC_i\\ 0 & C_N\end{bmatrix} \ge n + \operatorname{rank}[E]. \tag{22}$$

Using Lemma 2.3 again, the necessary and sufficient conditions for (21) and (22) to be true are that the following four inequalities are true:

$$\begin{aligned}
&\underset{K_1,\cdots,K_{N-2}}{\text{g.r.}}\begin{bmatrix}0 & E & 0 & 0\\ E & A+\sum_{i=1}^{N-2}B_iK_iC_i & B_{N-1} & B_N\end{bmatrix}\\
&\quad\ge n + \operatorname{rank}[E]
\end{aligned}\tag{23}$$

$$\begin{aligned}
&\underset{K_1,\cdots,K_{N-2}}{\text{g.r.}}\begin{bmatrix}0 & E & 0\\ E & A+\sum_{i=1}^{N-2}B_iK_iC_i & B_N\\ 0 & C_{N-1} & 0\end{bmatrix}\\
&\quad\ge n + \operatorname{rank}[E]
\end{aligned}\tag{24}$$

$$\begin{aligned}
&\underset{K_1,\cdots,K_{N-2}}{\text{g.r.}}\begin{bmatrix}0 & E & 0\\ E & A+\sum_{i=1}^{N-2}B_iK_iC_i & B_{N-1}\\ 0 & C_N & 0\end{bmatrix}\\
&\quad\ge n + \operatorname{rank}[E]
\end{aligned}\tag{25}$$

and

$$\underset{K_1,\cdots,K_{N-2}}{\text{g.r.}}\begin{bmatrix}0 & E \\ E & A+\sum_{i=1}^{N-2}B_iK_iC_i \\ 0 & C_{N-1} \\ 0 & C_N\end{bmatrix} \ge n+\text{rank}[E]. \tag{26}$$

Proceeding with this way, we can easily conclude that the necessary and sufficient condition for (20) to be true is for all $\varphi \in \mathcal{P}(\mathcal{N})$

$$\text{rank}\begin{bmatrix}0 & E & 0 \\ E & A & B_\varphi \\ 0 & C_{\mathcal{N}-\varphi} & 0\end{bmatrix} \ge n+\text{rank}[E]. \tag{27}$$

This completes the proof. □

Remark 3.1: The result given in [9] is only for centralized systems which is just a special case of Theorem 3.1 (i.e., $N = 1$). For a decentralized singular system, which is regular with impulsive modes, it is said to have decentralized impulse fixed modes (IFM's) if for any local output feedbacks $K \in \mathcal{S}_\mathcal{C}$, the following inequality holds:

$$\deg\left\{\det\left[sE - A + \sum_{i=1}^{N}B_iK_iC_i\right]\right\} < \text{rank}[E]. \tag{28}$$

Algebraic characterization of IFM's has been presented in [16]. The conditions given in Theorem 3.1 are the same in form as the conditions on nonexistence of the IFM's for regular systems. However, they will never be regarded as the same thing. The reason is that the conditions for IFM's are derived from the preassumption of regularity in the considered system.

Remark 3.2: The result in Theorem 3.1 is related to the structural controllability and structural observability of singular systems (see, e.g., [17]–[19]). Consider a class of structural singular systems described by

$$\begin{aligned} E\dot{x} &= A_px + Bu \\ y &= Cx \end{aligned} \tag{29}$$

where $E \in R^{n\times n}$ is singular, A_p is a parameterized matrix of the form

$$A_p = Q_0 + \sum_{i=1}^{N} M_iK_iN_i$$

Q_0, M_i, and N_i are constant matrices with appropriate sizes, respectively, K_i is a variable matrix with compatible dimension, and $i \in \mathcal{N}$.

Recall that matrix A_r is said to be a realization of A_p if it is obtained from some fixed parameters matrices K_i, $i \in \mathcal{N}$. System (29) is said to be structurally regular if there is a realization (E, A_r, B, C) of system (29) to be regular. Suppose that system (29) is structurally regular, then it is said to be structurally impulse controllable (observable), if there is a realization (E, A_r, B, C) of system (29) to be impulse controllable (observable) [19]. Thus, algebraic necessary and sufficient conditions on the structurally impulsive controllability and the structurally impulsive observability of system (29) can be obtained by using the same approach used in the proof of Theorem 3.1. The results are, respectively

$$\text{rank}\begin{bmatrix}0 & E & 0 & 0 \\ E & Q_0 & M_\varphi & B \\ 0 & N_{\mathcal{N}-\varphi} & 0 & 0\end{bmatrix} \ge n+\text{rank}[E] \tag{30}$$

and

$$\text{rank}\begin{bmatrix}0 & E & 0 \\ E & Q_0 & M_\varphi \\ 0 & N_{\mathcal{N}-\varphi} & 0 \\ 0 & C & 0\end{bmatrix} \ge n+\text{rank}[E] \tag{31}$$

for all $\varphi \in \mathcal{P}(\mathcal{N})$. Conditions (30) and (31) are the necessary and sufficient conditions for the existence of a static output feedback $u = Ky$, such that the closed-loop system $E\dot{x} = (A_p + BKC)x$ is structural impulse-free, i.e., there is a realization $(E, A_r + BKC)$ of the system $(E, A_p + BKC)$ to be regular and impulse-free. From Theorem 3.1, these conditions are equivalent to the conditions of the existence of decentralized control laws of the form $u_i = K_iy_i + v_i, i = 0, 1, \cdots, N$, that make the following system:

$$\begin{aligned} E\dot{x} &= Ax + \sum_{i=0}^{N} M_iu_i \\ y_i &= N_ix, \qquad i = 0, 1, \cdots, N \end{aligned} \tag{32}$$

both regular and impulse-free, where $M_0 = B$ and $N_0 = C$.

Now, we apply the following local P-D output feedback control laws to the singular system (1). That is

$$u_i = -L_i\dot{y}_i + K_iy_i + v_i, \qquad i = 1, 2, \cdots, N. \tag{33}$$

The closed-loop system will be of the following form:

$$\left[E + \sum_{i=1}^{N}B_iL_iC_i\right]\dot{x} = \left[A + \sum_{i=1}^{N}B_iK_iC_i\right]x + \sum_{i=1}^{N}B_iv_i. \tag{34}$$

The singular system (1) may be a normal system as $\text{g.r.}[E + \Sigma_{i=1}^{N}\ B_iL_iC_i] = n$. As $\text{g.r.}[E + \Sigma_{i=1}^{N}\ B_iL_iC_i] < n$, then system (34) is still a singular system. Applying the results above, we now derive the algebraic conditions for the existence of decentralized P-D output feedback control laws (33) that will make the closed-loop singular system (34) satisfy condition (2). In fact, the closed-loop system (34) will be regular and impulse-free as long as condition (2) holds.

Theorem 3.2: Given the singular system (1), there exists decentralized P-D control laws of the form $u_i = -L_i\dot{y}_i + K_iy_i + v_i, i = 1, 2, \cdots, N$, that make the closed-loop system (34) satisfy condition (2) if and only if

$$\text{rank}\begin{bmatrix}0 & E & 0 & B_{\varphi_1} & 0 \\ E & A & B_\varphi & 0 & B_{\varphi_2} \\ 0 & C_{\mathcal{N}-\varphi} & 0 & 0 & 0 \\ 0 & C_{\varphi-\varphi_1} & 0 & 0 & 0 \\ C_{(\mathcal{N}-\varphi)-\varphi_2} & 0 & 0 & 0 & 0\end{bmatrix} \ge n + R_g \tag{35}$$

for all $\varphi \in \mathcal{P}(\mathcal{N})$, $\varphi_1 \in \mathcal{P}(\varphi)$, and $\varphi_2 \in \mathcal{P}(\mathcal{N} - \varphi)$, with

$$R_g \triangleq \min_{\varphi\in\mathcal{P}(\mathcal{N})} \text{rank}\begin{bmatrix}E & B_\varphi \\ C_{\mathcal{N}-\varphi} & 0\end{bmatrix}. \tag{36}$$

Proof: By Lemmas 2.2 and 2.3, we know that the following holds for almost all $\tilde{L} \in \mathcal{S}_\mathcal{C}$:

$$\text{rank}\left[E + \sum_{i=1}^{N}B_i\tilde{L}_iC_i\right] = \underset{L_1,\cdots,L_N}{\text{g.r.}}\left[E + \sum_{i=1}^{N}B_iL_iC_i\right] = R_g. \tag{37}$$

In light of Theorem 3.1, condition (2) holds if and only if

$$\underset{L_1,\cdots,L_N}{\text{g.r.}}\begin{bmatrix}0 & E+\sum_{i=1}^{N}B_iL_iC_i & 0 \\ E+\sum_{i=1}^{N}B_iL_iC_i & A & B_\varphi \\ 0 & C_{\mathcal{N}-\varphi} & 0\end{bmatrix} \ge n + R_g \tag{38}$$

holds for all $\varphi \in \mathcal{P}(\mathcal{N})$. Observe that

$$\underset{L_1,\cdots,L_N}{\text{g.r.}} \begin{bmatrix} 0 & E + \sum_{i=1}^{N} B_i L_i C_i & 0 \\ E + \sum_{i=1}^{N} B_i L_i C_i & A & B_\varphi \\ 0 & C_{\mathcal{N}-\varphi} & 0 \end{bmatrix} = \underset{L_1,\cdots,L_N}{\text{g.r.}} \begin{bmatrix} 0 & E + \sum_{i\in\varphi} B_i L_i C_i & 0 \\ E + \sum_{i\in\mathcal{N}-\varphi} B_i L_i C_i & A & B_\varphi \\ 0 & C_{\mathcal{N}-\varphi} & 0 \end{bmatrix}. \tag{39}$$

Since $\varphi \cap \mathcal{N} - \varphi = \emptyset$ (empty set), then by using Lemma 2.3 on the right-hand side of (39) and repeating the procedure as shown in Theorem 3.1, we can finally obtain the algebraic conditions (35). □

Corollary 3.1: The derived algebraic conditions given in Theorem 3.2 for centralized systems become

$$\text{rank}\begin{bmatrix} 0 & E & 0 & B \\ E & A & B & 0 \end{bmatrix} \geq n + \min\left\{\text{rank}[E, B], \text{rank}\begin{bmatrix} E \\ C \end{bmatrix}\right\} \tag{40}$$

$$\text{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} \geq n + \min\left\{\text{rank}[E, B], \text{rank}\begin{bmatrix} E \\ C \end{bmatrix}\right\} \tag{41}$$

and

$$\text{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \\ C & 0 \end{bmatrix} \geq n + \min\left\{\text{rank}[E, B], \text{rank}\begin{bmatrix} E \\ C \end{bmatrix}\right\}. \tag{42}$$

Remark 3.3: A similar comment as in Remark 3.2 on the result stated in Theorem 3.2 can be made, which is related to the following structural singular system described by

$$\begin{aligned} E_p \dot{x} &= A_p x + Bu \\ y &= Cx \end{aligned} \tag{43}$$

where E_p and A_p are parameterized matrices of the form

$$E_p = P_0 + \sum_{i=1}^{n_1} M_i^1 K_i^1 N_i^1, \qquad A_p = Q_0 + \sum_{i=1}^{n_2} M_i^2 K_i^2 N_i^2$$

$P_0, Q_0, M_i^1, M_i^2, N_i^1$, and N_i^2 are constant matrices with appropriate sizes, respectively, K_i^1 and K_i^2 are variable matrices with compatible dimension, and $i \in \mathcal{N}$. For the limitation of space, the details are omitted here.

It should be pointed out that the results in this paper are based on the original system parameter matrices and no matrix manipulation and partitioning are required.

IV. Conclusion

This paper considers the problem of regularization of singular systems using decentralized output feedback. Algebraic necessary and sufficient conditions are derived which ensures the existence of decentralized output feedback control laws that will make the singular system both regular and impulse-free. Necessary and sufficient conditions are also given for the existence of P-D decentralized output feedback control laws, for which the closed-loop system will be regular and impulse-free with a maximal dynamical order.

Acknowledgment

Dr. D. Wang is grateful to Prof. X. K. Xie at Northeastern University, China, for his earlier guidance in studying singular systems. Thanks are also given to the anonymous referees for their helpful suggestions and valuable comments.

References

[1] E. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, vol. 26, pp. 702–707, 1981.

[2] S. L. Campbell, *Singular Systems of Differential Equations II*. New York: Pitman, 1982.

[3] L. Dai, *Singular Control Systems*, Lecture Notes in Control and Information Sciences. Berlin: Springer-Verlag, 1989.

[4] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. 26, pp. 811–831, 1981.

[5] F. L. Lewis, "A survey of linear singular systems," *Circuit Syst. Signal Processing*, vol. 5, pp. 3–36, 1986.

[6] K. Ozcaldiran and F. L. Lewis, "On the regularizability of singular systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1156–1160, 1990.

[7] L. R. Fletcher, "Regularizability of descriptor systems," *Int. J. Syst. Sci.*, vol. 47, pp. 843–847, 1986.

[8] V. Lovass-Nagy, D. L. Powers, and R. J. Schilling, "A note on regularizing descriptor-variable systems by proportional-plus derivative feedback," *J. Math. Syst. Estim. Contr.*, vol. 4, pp. 341–351, 1994.

[9] ——, "On regularizing descriptor systems by output feedback," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1507–1509, 1994.

[10] A. Bunse-Gerstner, V. Mehrmann, and N. K. Nichols, "Regularization of descriptor systems by output feedback," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1742–1748, 1994.

[11] L. R. Fletcher, J. Kautsky, and N. K. Nichols, "Eigenstructure assignment in descriptor systems," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 1138–1141, 1986.

[12] V. Kucera and P. Zagalak, "Constant solutions of polynomial equations," *Int. J. Contr.*, vol. 53, pp. 495–502, 1991.

[13] J. Kautsky, N. K. Nichols, and K. W. E. Chu, "Robust pole assignment in singular control systems," *Linear Algebra and Its Appl.*, vol. 121, pp. 9–37, 1989.

[14] J. M. Potter, B. D. O. Anderson, and A. S. Morse, "Single-channel control of a two-channel systems," *IEEE Trans. Automat. Contr.*, vol. 24, pp. 491–492, 1979.

[15] X. K. Xie and Y. Z. Yang, "Frequency domain characterization of decentralized fixed modes," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 952–955, 1986.

[16] X. K. Xie, D. Wang, C. Lin, and X. J. Cheng, "A unified approach to study fixed modes in singular systems," *Acta Automatica Sinica*, vol. 21, pp. 153–164, 1995.

[17] T. Yamada and D. G. Luenberger, "Generic controllability theorems for descriptor systems," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 144–152, 1985.

[18] K. Murota, "Refined study on structural controllability of descriptor systems by means of matroids," *SIAM J. Contr. Optim.*, vol. 25, pp. 967–989, 1987.

[19] Q. L. Zhang, "Structurally impulsive controllability for descriptor systems," in *Proc. American Control Conf.*, 1990, vol. 2, pp. 2203–2206.